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On the Wavelet Analysis for Multifractal Sets

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We establish a rigorous relation between the wavelet transform of a measure and its local scaling exponents.

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The wavelet transform has been introduced and developed in the last few years[1–3] to study a large class of phenomena. Although very different, they have the common feature of showing a scale invariance. In particular, the wavelet transform has been recently applied to the invariant sets under smooth mappings in order to analyze the multifractal properties of the measures supported by them.[4–6] In this note we prove a rigorous result in this direction, which also explains the numerical results and the illustrations quoted in refs. 4–6. Let J be a subset of the metric space Ω, with metric $\|\cdot\|$, which supports a probability Borel measure μ. We recall that the wavelet transform of the measure μ is defined as[4–6]

$$T_p(a, b) = \frac{1}{a^p} \int_J g\left(\frac{\|x-b\|}{a}\right) d\mu(x) \tag{1}$$

where a and $p > 0$, $b \in J$, J has a finite diameter L, and g: $\mathbb{R} \to \mathbb{R}$ satisfies:

(i) g is C^1 on $\mathbb{R}$.

(ii) $\lim_{a \to 0^+} a^{-p} g(r/a) = 0$ pointwise for $r \geq 0$ and for any $p > 0$.

(iii) $g'(r) < 0$ for $r \in (0, \alpha)$; $g'(r) > 0$ for $r \in (\alpha, +\infty)$; and $r^\gamma g'(r)$ is summable on $[0, +\infty)$ for any $\gamma > -1$.

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A possible choice for $g(r)$ is

$$g(r) = (d - r^2)e^{-r^2/2} \tag{2}$$

where d is the dimension of Ω, if it has a manifold structure. This wavelet is also called the "Mexican hat" and is used in the numerical computations of $T_p(a, b)$.

The above conditions are sufficient for the following. Of course, they can be weakened [for example, one could require that $g(r)$ is of bounded variation on $\mathbb{R}$, with a fast decay at infinity], but this is not really important in view of the numerical computations of $T_p(a, b)$ (see also Remark 2 below).

On the other hand, we want to point out that the proof of the theorem below does not need some usual properties of the wavelets, for instance, the fact that $g(r)$ is of zero mean on $\mathbb{R}$ with respect to the Lebesgue measure.

In refs. 4–6, the following nonrigorous scaling for the wavelet transform has been proposed:

$$T_p(\lambda a, x_0 + \lambda b) \underset{\lambda \downarrow 0^+}{\sim} \lambda^{(\alpha(x_0) - p} T_p(a, x_0 + b) \tag{3}$$

where $\lambda > 0$, x_0 and $b \in J \subset \Omega$ (which is also supposed to be a vector space), and $\alpha(x_0)$ is the local scaling index of the point x_0 defined as

$$\mu(B(x_0, l)) \underset{l \downarrow 0^+}{\sim} l^{\alpha(x_0)}$$

$B(x_0, l)$ is a ball of center x_0 and radius l. It is clear from (3) that the wavelet transform has a transition point from zero to infinity for $p = \alpha(x_0)$ and this point is numerically computable. So one has a very powerful tool to locate and evaluate the singularities of the measure μ, whose only density was obtained by means of the α–$f(\alpha)$ spectrum (ref. 7 and Remark 1 below). The rigorous statement is a little bit different from (3).

Theorem. Let J be a subset of Ω which supports a probability Borel nonatomic measure μ and set for any point $b \in J$

$$\beta_- = \liminf_{l \to 0^+} \frac{\log[\mu(B(b, l))]}{\log l} \leqslant \limsup_{l \to 0^+} \frac{\log[\mu(B(b, l))]}{\log l} = \beta_+ \tag{4}$$

Assume $\beta_+ < +\infty$. Then

$$\inf\{p; \limsup_{a \to 0^+} |T_p(a, b)| > 0\} \geqslant \beta_- \tag{5}$$

Remark 1. For a large class of dynamical systems the limit (4) exists μ-almost everywhere and gives the so-called Hausdorff dimension of the measure $HD(\mu)=\beta_+=\beta_-$, which is also related to some dynamical indices such as the metric entropy and the Lyapynov exponents of μ.[8]

We also recall that, for a large class of expanding dynamical systems, it is possible to compute the Hausdorff dimension of the set of points for which the lim sup (resp. lim inf) is the same in (4): this leads to the so-called α–$f(\alpha)$ theory, for which we refer to ref. 9 for rigorous results.

Remark 2. The numerical computations of $T_p(a, b)$ for multifractal sets[4 6,12] clearly show that the infimum of p given by (5) is finite. We want to remark that for a particular choice of the wavelet this infimum could be infinite.

For example, let us consider the unit segment $[0, 1] \subset \mathbb{R}$, endowed with the Lebesgue measure, and consider the transform of the Mexican hat (2). A straightforward integration gives for each $b \in [0, 1]$

$$T_p(a, b)=\frac{1}{a^p}\left\{(1-b)\exp\left[-\frac{1}{2}\left(\frac{1-b}{a}\right)^2\right]+b\exp\left[-\frac{1}{2}\left(\frac{b}{a}\right)^2\right]\right\}$$

which is zero for any finite, positive p in the limit $a \to 0^+$.

This is easily seen to occur when the measure has a density which is dominated by the faster decay of the wavelet at infinity given by condition (ii) above. This is not the case, in general, for the measures supported by fractal sets which can be obtained as the weak limit of Dirac measures. However, the problem of adapting a wavelet to a measure (and vice versa) in order to find a finite transition point in (5) is open and very interesting (see refs. 10 and 11 for some rigorous results in this direction).

Proof of the Theorem. For any positive δ there exists an $l_\delta < 1$ such that for $l < l_\delta$ one has

$$l^{\beta_+ +\delta} \leqslant \mu(B(b, l)) \leqslant l^{\beta_- -\delta} \tag{6}$$

We introduce the Borel measure, supported by a closed subset of $[0, L]$, L being the diameter of J:

$$m_b(r)=\mu((B(b, r)) \cap J), \qquad r \geqslant 0$$

and rewrite (1) as

$$T_p(a, b)=\frac{1}{a^p}\int_0^L g\left(\frac{r}{a}\right) dm_b(r)$$

Making the change of variable $r=r'a$ (and setting again $r'=r$) and integrating by parts, we get

$$T_p(a, b)=\frac{1}{a^p}\left[g(r)\, m_b(ra)\right]_0^{L/a}-\frac{1}{a^p}\int_0^{L/a} g'(r)\, m_b(ra)\, dr \tag{7}$$

Now, since $m_b(L) = 1$, the first term in the rhs of (7) is simply $a^{-p} g(L/a)$ and it converges to zero when $a \to 0^+$, by the assumption (ii) on the wavelet. So we neglect this term in the following since it does not contribute to the final result. Then we rewrite the second term in the rhs of (7) as the sum of three pieces:

$$-a^{-p} \int_0^{\alpha} g'(r)\, m_b(ra)\, dr - a^{-p} \int_{\alpha}^{l_\delta/a} g'(r)\, m_b(ra)\, dr$$
$$-a^{-p} \int_{l_\delta/a}^{L/a} g'(r)\, m_b(ra)\, dr \tag{8}$$

having chosen $a < l_\delta/\alpha$.

The last term in (8) simply gives

$$0 < a^{-p} \int_{l_\delta/a}^{L/a} g'(r)\, m_b(ra)\, dr \leqslant a^{-p} [g(r)]_{l_\delta/a}^{L/a}$$

and, as before, it converges to zero, so that we neglect it. Now we consider the second term in (8). By assumption (iii), $g'(r)$ is positive for $r \in (\alpha, l_\delta/a]$. Then, using the bound (6), we get

$$\begin{aligned} -a^{-p} \int_{\alpha}^{l_\delta/a} g'(r)\, m_b(ra)\, dr &\leqslant -a^{-p+(\beta_+ +\delta)} \alpha^{(\beta_+ +\delta)} [g(r)]_{\alpha}^{l_\delta/a} \\ &= a^{-p+(\beta_+ +\delta)} \alpha^{(\beta_+ +\delta)} g(\alpha) \\ &\quad - a^{-p+(\beta_+ +\delta)} \alpha^{(\beta_+ +\delta)} g(l_\delta/a) \end{aligned} \tag{9}$$

In the following we neglect the second term in the rhs of (9), since it vanishes in the limit $a \to 0^+$.

In the same way we have

$$-a^{-p} \int_{\alpha}^{l_\delta/a} g'(r)\, m_b(ra)\, dr \geqslant -a^{-p+(\beta_- -\delta)} \int_{\alpha}^{\infty} g'(r)\, r^{\beta_- -\delta}\, dr \tag{10}$$

where the integral on the rhs is finite by the assumption (iii) on the wavelet and taking $\delta < 1$ (in view of the possible value $\beta_- = 0$).

Finally, the first term in (8) is easily bounded in the same way as

$$-a^{-p} \int_0^{\alpha} g'(r)\, m_b(ra)\, dr \leqslant -a^{-p+(\beta_- -\delta)} \int_0^{\alpha} g'(r)\, r^{\beta_- -\delta}\, dr \tag{11}$$

and

$$-a^{-p} \int_0^{\alpha} g'(r)\, m_b(ra)\, dr \geqslant -a^{-p+(\beta_+ +\delta)} \int_0^{\alpha} g'(r)\, r^{\beta_+ +\delta}\, dr \tag{12}$$

Collecting (9)–(12), we get

$$T_p(a, b) \leqslant a^{-p+(\beta_- - \delta)} \left[a^{2\delta + (\beta_+ - \beta_-)} \alpha^{\beta_+ + \delta} g(\alpha) - \int_0^\alpha g'(r)\, r^{\beta_- - \delta}\, dr \right] \quad (13)$$

and

$$T_p(a, b) \geqslant a^{-p+(\beta_- - \delta)} \times \left[-a^{2\delta + (\beta_+ - \beta_-)} \int_0^\alpha g'(r)\, r^{\beta_+ + \delta}\, dr - \int_\alpha^\infty g'(r)\, r^{\beta_- - \delta}\, dr \right] \quad (14)$$

When $a \to 0^+$, the first term into brackets of (13) and (14) tends to zero, so we get

$$\inf\{p;\ \limsup_{a \to 0^+} |T_p(a, b)| > 0\} \geqslant \beta_- - \delta$$

which gives the theorem, δ being arbitrary. ▮

We incidentally note that assuming for any point $x_0 \in J$ a scaling of the type

$$\mu(B(x_0, l)) \sim l^{\alpha(x_0)}$$

a formal application of the above arguments gives the (nonrigorous) scaling

$$T_p(\lambda a, b) \underset{\lambda \downarrow 0^+}{\sim} \lambda^{\alpha(b) - p} T_p(a, b) \quad (15)$$

which is different from (3), being more related to the fractal properties of the set than to the invariance properties of the wavelet. However, the two scalings can be numerically studied in the same way.[12]

We note that this formula is obtained under the hypothesis that $\beta_- = \beta_+ = \alpha(b)$ in (4) and the wavelet transform of μ has a transition point exactly for $p = \alpha(b)$. This last fact could be true for the invariant sets with good hyperbolic properties as preliminary numerical investigations seem to suggest, but nevertheless it is not easy to prove [for example, the oscillations exhibited by $T_p(a, b)$ when $a \to 0^+$ could prevent the existence of the limit in relation (5)].

We finally point out that, taking p in (1) greater than or equal to the Hausdorff dimension of the set (which is generally greater than any local scaling index β), the relation (5) implies that the wavelet transform $|T_p(a, b)|$ will be in general different from zero when $a \to 0^+$, and this explains the nice pictures obtained in refs. 4–6 and 12 by applying the scalings (3) and (15).

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